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Self-propulsion of oscillating wings in incompressible flow

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SUMMARY

In this paper, we show that the oscillatory motion of an airfoil (wing) in an ideal fluid can determine the apparition of thrust. In the framework of the linearized perturbation theory, the pressure jump over the oscillating wing is the solution of a two-dimensional hypersingular integral equation. Using appropriate quadrature formulas, we discretize the oscillatory lifting surface integral equation in order to obtain the jump of the pressure across the surface. Integrating numerically, we obtain the drag coefficient. For some oscillatory motions, if the frequency of the oscillations surpasses a certain value, the drag coefficient becomes negative, i.e. there appears a propulsive force. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The thrust caused by the oscillatory or undullatory motion of rigid or flexible thin airfoils is a very important aerodynamic phenomenon with many engineering applications to microvehicles motion. Therefore, a great number of papers were devoted to the applications of computational fluid mechanics to this topic. In our paper, we utilize the lifting surface theory for investigating the incompressible flow past a wing performing an oscillatory (or undullatory) motion. As it is known, this theory furnishes good results for thin airfoils and small angles of attack. In exchange for the limitations concerning the geometry of the airfoil, the lifting surface theory has a nice mathematical formulation which leads to an integral equation for the jump of the pressure across the airfoil.

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The problem of the oscillatory thin wings in subsonic flow (and implicitly in incompressible flow) was studied, among others by Watkins *et al.* [1], Laschka [2], and Landahl [3]. In their theory, the integral equation, originally performed by Küssner [4] is obtained by determining the acceleration potential due to a continuous distribution of doublets on the wing.

Later, Dragoş [5] studied the problem of oscillating wings by means of the fundamental solution method. Homentcovschi [6] utilized the Fourier transform for obtaining the fundamental solutions of the linear Euler system and then the general integral equation relating the jump of the pressure and the downwash distributions for the unsteady flow past a lifting surface, moving on a cylindrical surface. From this equation, it was deduced the integral equation for the oscillating wing.

Another method for deducing the fundamental solutions of the linear Euler system and then the integral equation of the oscillating wing is presented in Appendix A of this paper.

Many numerical methods were developed for solving the integral equation of steady or oscillatory lifting surface equation. Among them, we mention the methods considered by Ichikawa [7], Ueda and Dowell [8], Eversman and Pitt [9], etc.

The great number of papers devoted to the numerical methods used for integrating the lifting surface integral equation is justified by the difficulties caused by the singularities of the kernel. In our paper, in order to discretize the integral equation, we split the kernel of the equation into several kernels for which we provide appropriate quadrature formulas depending on the type of singularity of the kernel. By solving the discretized integral equation, we calculate the jump of the pressure over the wing.

After obtaining the pressure field we calculate, by performing a numerical integration, the average drag. We study an example of oscillatory motion of a flexible delta wing and we notice that if the reduced frequency surpasses a critical value, the drag becomes negative, i.e. it appears a propulsive force. The numerical method we present herein is valid for all thin airfoils of arbitrary aspect ratio. In order to compare the numerical results with analytical ones, we consider the delta wing of low aspect ratio. Performing an asymptotic expansion of the kernel of the integral equation with respect to ϖ (the aspect ratio) and neglecting the terms of order $O(\varpi^2 \ln \varpi)$, one obtains instead of a two-dimensional integral equation a one-dimensional hypersingular integral equation (for the jump of the pressure) which is solved analytically. The one-dimensional equation is an equation which approximates the original one; we deduce therefore that the analytical results are approximate results and the order of the asymptotic approximation is $O(\varpi \ln \varpi)$. A comparison between the analytical and numerical results for the symmetrical delta wing of low aspect ratio shows a very good agreement.

2. THE INTEGRAL EQUATION

We consider a system of coordinates $Ox^{(1)}y^{(1)}z^{(1)}$ related to the wing, and we introduce the dimensionless space coordinates $(x, z) = (x^{(1)}/a, z^{(1)}/a)$, taking the wing length (more precisely the maximum value of the wing chord) a as reference length along the vertical direction ($Oz^{(1)}$-axis direction) and along the direction of the unperturbed uniform flow ($Ox^{(1)}$-axis direction). We introduce also the dimensionless space coordinate $y = y^{(1)}/b$, taking the wing half-span b as reference length along the $Oy^{(1)}$-axis direction. Denote by $D^{(1)}$ the wing projection on the $Ox^{(1)}y^{(1)}$-plane. Let

$$0 = F(x^{(1)}, y^{(1)}, z^{(1)}, t) = z^{(1)} - h^{(1)}(x^{(1)}, y^{(1)}) \exp(\mathrm{i}\omega t), \quad (x^{(1)}, y^{(1)}) \in D^{(1)} \tag{1}$$

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be the equation of the wing (we neglect the thickness of the wing). We notice that the coordinates $(x^{(1)}, y^{(1)})$ of every point of the wing remain unchanged and the coordinate $z^{(1)}$ varies harmonically. Throughout this paper, depending on the context (like in many other papers dedicated to the oscillatory flow of a fluid), when we encounter a complex number or function, we have in mind its real part.

Let $\rho_0 Re(f \exp(\mathrm{i}\omega t))$ be the jump of the pressure over the oscillating wing. ω is the frequency of the oscillation, V_0 is the translation velocity of the unperturbed flow with respect to the $Ox^{(1)}y^{(1)}z^{(1)}$ frame of reference and $\varpi = b/a$ is the aspect ratio. Introducing the dimensionless functions and variables

$$h(x, y) = \frac{h^{(1)}(x^{(1)}, y^{(1)})}{a}, \quad \tilde{\omega} = \frac{\omega a}{V_0}, \quad \tilde{f}(x, y) = \frac{f(ax, by)}{V_0^2}$$

$$x_0 = x - \xi, \quad y_0 = y - \eta$$

one demonstrates [6, 10] that the integral equation of the oscillatory thin wing is

$$\frac{-\varpi}{4\pi} \iint_D^* \tilde{f}(\xi, \eta) \exp(-\mathrm{i}\tilde{\omega} x_0) \left(\int_{-\infty}^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega} s)}{(s^2 + \varpi^2 y_0^2)^{3/2}} \, \mathrm{d}s \right) \mathrm{d}\xi \, \mathrm{d}\eta$$

$$= \frac{\partial h(x, y)}{\partial x} + \mathrm{i}\tilde{\omega} h(x, y) \tag{2}$$

where $(x, y) \in D$ if and only if $(x^{(1)}, y^{(1)}) \in D^{(1)}$.

The asterisk indicates the finite part in the Hadamard sense of the integral.

3. THE DISCRETIZATION OF THE INTEGRAL EQUATION FOR THE SYMMETRICAL OSCILLATING DELTA FLAT PLATE

We consider the oscillating delta wing. The equations of the leading edge of $D^{(1)}$ are

$$y_{\pm}^{(1)}(x^{(1)}) = \pm \frac{b}{a} (x^{(1)}), \quad x^{(1)} \in [0, a] \tag{3}$$

and the equations of the leading edge of D are

$$y_{\pm}(x) = \pm x, \quad x \in [0, 1] \tag{4}$$

For solving numerically the integral equation (2), we have to discretize the left-hand member in order to obtain an algebraic system of equations. To this aim, we split the kernel $K(x, y; \xi, \eta) = \int_{-\infty}^{x_0} [\exp(\mathrm{i}\tilde{\omega} s)/(s^2 + \varpi^2 y_0^2)^{3/2}] \, \mathrm{d}s$ into several kernels in order to put into evidence the kind of singularities we are dealing with and to find afterwards the most suitable quadrature formulas.

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We have step by step

$$\int_{-\infty}^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = \int_{-\infty}^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s + \frac{1}{\varpi^2 y_0^2}\left(1+\frac{x_0}{|x_0|}\right) + \frac{1}{\varpi^2 y_0^2}\left(\frac{x_0}{\sqrt{x_0^2+\varpi^2 y_0^2}} - \frac{x_0}{|x_0|}\right) \tag{5}$$

$$\int_{-\infty}^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}u = \int_0^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s + \int_0^{\infty} \frac{\exp(-\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s \tag{6}$$

$$\int_0^{\infty} \frac{\exp(-\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = -\frac{1}{\varpi^2 y_0^2} + \int_0^{\infty} \frac{\cos\tilde{\omega}s}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s - \mathrm{i}\int_0^{\infty} \frac{\sin\tilde{\omega}s}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s \tag{7}$$

The integrals from the right-hand part of (7) represent the *sine* and *cosine Fourier transforms* of $(s^2+\varpi^2 y_0^2)^{-3/2}$ and in [11] one shows that

$$\int_0^{\infty} \frac{\cos\tilde{\omega}s}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = \frac{\tilde{\omega}}{\varpi|y_0|} K_1(\tilde{\omega}\varpi|y_0|) \tag{8}$$

$$\int_0^{\infty} \frac{\sin\tilde{\omega}s}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = \frac{\pi}{2}\frac{\tilde{\omega}}{\varpi|y_0|}(L_{-1}(\tilde{\omega}\varpi|y_0|) - I_1(\tilde{\omega}\varpi|y_0|)) \tag{9}$$

where L_{-1} is a Strouve function, I_1, K_1 are Bessel functions and the series expansions are

$$I_1(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!} \tag{10}$$

$$K_1(x) = I_1(x)\ln\frac{x}{2} + \frac{1}{x} - \sum_{k=0}^{\infty} \frac{(x/2)^{2k+1}}{k!(k+1)!}(\psi(k+1)+\psi(k+2)) \tag{11}$$

$$L_{-1}(x) = \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{\Gamma(k+\frac{3}{2})\Gamma(k+\frac{1}{2})} \tag{12}$$

where ψ represents the logarithmic derivative of Euler's Γ function.

We also have

$$\int_0^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)-1}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = \int_0^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)-1-\mathrm{i}\tilde{\omega}s+\tilde{\omega}^2 s^2/2}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s - \frac{\mathrm{i}\tilde{\omega}}{(x_0^2+\varpi^2 y_0^2)^{1/2}} + \frac{\mathrm{i}\tilde{\omega}}{|y_0|} + \frac{\tilde{\omega}^2 x_0}{2(x_0^2+\varpi^2 y_0^2)^{1/2}} - \frac{\tilde{\omega}^2}{2}\ln(x_0+\sqrt{(x_0{}^2+\varpi^2 y_0^2)}) \tag{13}$$

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Hence

$$K(x, y; \xi, \eta) = K_1(x, y; \xi, \eta) + \cdots + K_8(x, y; \xi, \eta)$$

and the integral equation (2) becomes

$$\frac{\varpi}{4\pi}\sum_{i=1}^{8} \iint_D^* \tilde{f}(\xi, \eta) \exp(\mathrm{i}\tilde{\omega}\xi) K_i(x, y; \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta$$

$$= -\left(\frac{\partial h(x, y)}{\partial x} + \mathrm{i}\tilde{\omega} h(x, y)\right) \exp(\mathrm{i}\tilde{\omega} x) \tag{14}$$

Hence

$$K_1(x, y; \xi, \eta) = \frac{1}{\varpi^2 y_0^2}\left(\frac{x_0}{\sqrt{x_0^2 + \varpi^2 y_0^2}} - \frac{x_0}{|x_0|}\right) \tag{15}$$

$$K_2(x, y; \xi, \eta) = \frac{1}{\varpi^2 y_0^2}\left(1 + \frac{x_0}{|x_0|}\right) \tag{16}$$

$$K_3(x, y; \xi, \eta) = \frac{-\mathrm{i}\widetilde{\omega}}{\sqrt{x_0^2 + \varpi^2 y_0^2}} \tag{17}$$

$$K_4(x, y; \xi, \eta) = -\frac{\widetilde{\omega}^2}{2}\frac{x_0}{|x_0|} \ln\left(|x_0| + \sqrt{x_0^2 + \varpi^2 y_0^2}\right) \tag{18}$$

$$K_5(x, y; \xi, \eta) = \frac{\widetilde{\omega}^2}{2} \ln(\varpi |y_0|)\left(1 + \frac{x_0}{|x_0|}\right) \tag{19}$$

$$K_6(x, y; \xi, \eta) = \frac{\widetilde{\omega}^2 x_0}{\sqrt{x_0^2 + \varpi^2 y_0^2}} \tag{20}$$

$$K_7(x, y; \xi, \eta) = \frac{\widetilde{\omega}}{\varpi |y_0|} K_1(\widetilde{\omega}\varpi |y_0|) - \frac{1}{\varpi^2 y_0^2} - \frac{\widetilde{\omega}^2}{2} \ln \frac{\widetilde{\omega}\varpi |y_0|}{2} \tag{21}$$

$$+ \frac{\mathrm{i}\pi\widetilde{\omega}^2}{2\varpi |y_0|}\left(I_1(\widetilde{\omega}\varpi |y_0|) - L_{-1}(\widetilde{\omega}\varpi |y_0|) + \frac{2}{\pi}\right) + \frac{\widetilde{\omega}^2}{2} \ln \frac{\varpi}{2} \tag{22}$$

$$K_8(x, y; \xi, \eta) = \int_0^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega} s) - 1 - \mathrm{i}\tilde{\omega} s + \tilde{\omega}^2 s^2/2}{(s^2 + \varpi^2 y_0^2)^{3/2}}\, \mathrm{d}s \tag{23}$$

We shall provide adequate quadrature formulas for the integrals from the left-hand part of Equation (14) in order to discretize it. The analytical results from the sixth section suggest us

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to presume the following behaviour of the unknown function:

$$\tilde{f}(\xi,\eta)=\frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}}$$

whence we have

$$\begin{aligned}&\iint_D^* \tilde{f}(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)K_1(x,y;\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta\\&\quad=\frac{1}{\varpi^2}\mathrm{FP}\int_{-1}^{1}\frac{1}{y_0^2}\left(\int_{|\eta|}^{1}\frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}}\exp(\mathrm{i}\tilde{\omega}\xi)\left(\frac{x_0}{\sqrt{x_0^2+\varpi^2y_0^2}}-\frac{x_0}{|x_0|}\right)\mathrm{d}\xi\right)\mathrm{d}\eta\end{aligned}\tag{24}$$

where FP stands for the finite part of the hypersingular integral as it is introduced by Fox [12]. Taking into account that the integral from the right-hand side vanishes for $|\eta|=1$, we assume the following behaviour:

$$\begin{aligned}&\int_{|\eta|}^{1}\frac{g(\xi,\eta)}{\sqrt{\xi^2-\eta^2}}\exp(\mathrm{i}\tilde{\omega}\xi)\left(\frac{x_0}{\sqrt{x_0^2+\varpi^2y_0^2}}-\frac{x_0}{|x_0|}\right)\mathrm{d}\xi\\&\quad=\sqrt{1-\eta^2}G(x,y;\eta)\end{aligned}\tag{25}$$

where $G(x,y;\eta)$ is finite in $\eta=\pm1$. We consider on D a net consisting of the nodes (grid points, control points) $(x_i,\overline{y}_j)=(i/n,(2j+1)/2n)$, $i=1,\ldots,n$, $j=-i,-i+1,\ldots,i-1$. For the hypersingular integral occurring in (24) we may use the quadrature formula for equidistant control points given by Dumitrescu [13]:

$$\mathrm{FP}\int_{-1}^{1}\frac{\sqrt{1-\eta^2}G(x_k,\overline{y}_l;\eta)}{(\overline{y}_l-\eta)^2}\,\mathrm{d}\eta=\sum_{j=-n}^{n-1}G(x_k,\overline{y}_l;\overline{y}_j)A_{lj}\tag{26}$$

$$\begin{aligned}A_{lj}&=-\arccos(y_j)+\arccos(y_{j+1})+\frac{\sqrt{1-y_j^2}}{y_j-\overline{y}_l}\\&\quad-\frac{\sqrt{1-y_{j+1}^2}}{y_{j+1}-\overline{y}_l}-\frac{\overline{y}_l}{\sqrt{1-\overline{y}_l^2}}\ln\left|\frac{C_{l(j+1)}}{C_{lj}}\right|\end{aligned}\tag{27}$$

with

$$C_{lj}=\frac{\sqrt{1-y_j}\cdot\sqrt{1+\overline{y}_l}-\sqrt{1+y_j}\cdot\sqrt{1-\overline{y}_l}}{\sqrt{1-y_j}\cdot\sqrt{1+\overline{y}_l}+\sqrt{1+y_j}\cdot\sqrt{1-\overline{y}_l}}\tag{28}$$

DOI: 10.1002/fld

We have the quadrature formula

$$G(x_k,\overline{y}_l;\overline{y}_j)=\sum_{i=j}^{n} g_{ij}B_{ijkl} \tag{29}$$

with

$$g_{ij}=g(\overline{x}_{ij},\overline{y}_j)$$

$$\overline{x}_{ij}=\begin{cases} x_i-\dfrac{1}{2n}, & -i<j<i-1\\[2mm] x_i-\dfrac{1}{4n}, & j\in\{-i,i-1\}\end{cases},\quad \widetilde{x}_{ij}=\begin{cases} x_i-\dfrac{1}{n}, & -i<j<i-1\\[2mm] x_i-\dfrac{1}{2n}, & j\in\{-i,i-1\}\end{cases} \tag{30}$$

$$B_{ijk}=\frac{E_{ij}D_{ijkl}}{\sqrt{1-\overline{y}_j^2}} \tag{31}$$

$$E_{ij}=\exp(\mathrm{i}\tilde{\omega}\overline{x}_{ij})\left[\ln\left(x_i+\sqrt{x_i^2-\overline{y}_j^2}\right)-\ln\left(\overline{x}_{ij}+\sqrt{\widetilde{x}_{ij}^2-\overline{y}_j^2}\right)\right] \tag{32}$$

$$D_{ijkl}=\left(\frac{x_k-\overline{x}_{ij}}{\sqrt{(x_k-\overline{x}_{ij})^2+\varpi^2(\overline{y}_l-\overline{y}_j)^2}}-\frac{x_k-\overline{x}_{ij}}{|x_k-\overline{x}_{ij}|}\right),\quad -i<j<i-1 \tag{33}$$

Finally, we deduce

$$\iint_D^* \tilde{f}(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)K_1(x_k,\overline{y}_l;\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta=\sum_{i=1}^{n}\sum_{j=-i}^{i-1} g_{ij}K_{ijkl}^{(1)} \tag{34}$$

where

$$K_{ijkl}^{(1)}=\frac{A_{lj}B_{ijkl}}{\varpi^2} \tag{35}$$

Let

$$K_2(x,y;\xi,\eta)=\frac{1}{\varpi^2 y_0^2}\left(1+\frac{x_0}{|x_0|}\right) \tag{36}$$

We have

$$\begin{aligned}&\iint_D^* \tilde{f}(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)K_2(x,y;\xi,\eta)\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad=\frac{2}{\varpi^2}\mathrm{FP}\int_{-x}^{x}\frac{1}{y_0^2}\left(\int_{|\eta|}^{x}\frac{g(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)}{\sqrt{\xi^2-\eta^2}}\,\mathrm{d}\xi\right)\mathrm{d}\eta\end{aligned} \tag{37}$$

Since the integral inside the brackets vanishes for $x=|\eta|$, we may assume the behaviour

$$\int_{|\eta|}^{x_k}\frac{g(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)}{\sqrt{\xi^2-\eta^2}}\,\mathrm{d}\xi=\sqrt{x_k^2-\eta^2}\,G^{(k)}(x_k;\eta) \tag{38}$$

DOI: 10.1002/fld

where $G^{(k)}(x_k;\eta)$ is finite for $x_k=|\eta|$. We have therefore

$$\iint_D^* \tilde{f}(\xi,\eta)\exp(\mathrm{i}\tilde{\omega}\xi)K_2(x_k,\overline{y}_l;\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=\frac{2}{\varpi^2}\mathrm{FP}\int_{-x_k}^{x_k}\frac{\sqrt{x_k^2-\eta^2}G^{(k)}(x_k;\eta)}{(\overline{y}_l-\eta)^2}\,\mathrm{d}\eta=\frac{2}{\varpi^2}\sum_{j=-k}^{k-1}G^{(k)}(x_k;\overline{y}_j)A_{lj}^{(k)} \tag{39}$$

where

$$A_{lj}^{(k)}=-\arccos\left(\frac{y_j}{x_k}\right)+\arccos\left(\frac{y_{j+1}}{x_k}\right)$$

$$+\frac{\sqrt{x_k^2-y_j^2}}{y_j-\overline{y}_l}-\frac{\sqrt{x_k^2-y_{j+1}^2}}{y_{j+1}-\overline{y}_l}-\frac{\overline{y}_l}{\sqrt{x_k^2-\overline{y}_l^2}}\ln\left|\frac{C_{l(j+1)}^{(k)}}{C_{lj}^{(k)}}\right| \tag{40}$$

with

$$C_{lj}^{(k)}=\frac{\sqrt{x_k-y_j}\cdot\sqrt{x_k+\overline{y}_l}-\sqrt{x_k+y_j}\cdot\sqrt{x_k-\overline{y}_l}}{\sqrt{x_k-y_j}\cdot\sqrt{x_k+\overline{y}_l}+\sqrt{x_k+y_j}\cdot\sqrt{x_k-\overline{y}_l}} \tag{41}$$

For calculating $G^{(k)}(x_k;\overline{y}_j)$, we employ the quadrature formula

$$G^{(k)}(x_k;\overline{y}_j)=\frac{1}{\sqrt{x_k^2-\overline{y}_j^2}}\int_{|\overline{y}_j|}^{x_k}\frac{g(\xi,\overline{y}_j)\exp(\mathrm{i}\tilde{\omega}\xi)}{\sqrt{\xi^2-\overline{y}_j^2}}\,\mathrm{d}\xi=\sum_{i=|j|}^{k}g_{ij}\frac{E_{ij}}{\sqrt{x_k^2-\overline{y}_l^2}} \tag{42}$$

At last we find

$$\iint_D^* \tilde{f}(\xi,\eta)\ \exp(\mathrm{i}\tilde{\omega}\xi)K_2(x_k,\overline{y}_l;\xi,\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta=\sum_{i=1}^{n}\sum_{j=-i}^{i-1}g_{ij}K_{ijkl}^{(2)} \tag{43}$$

with

$$K_{ijkl}^{(2)}=\begin{cases}\dfrac{2}{\varpi^2}A_{lj}^{(k)}\dfrac{E_{ij}}{\sqrt{x_k^2-\overline{y}_l^2}}, & i\leqslant k\\[2ex] 0, \quad i>k\end{cases} \tag{44}$$

The kernels K_3 and K_4 are singular. We divide and multiply them by y_0^2 for obtaining quadrature formulas similar to the formulas for K_1. We get

$$\iint_D^* \tilde{f}(\xi,\eta)\ \exp(\mathrm{i}\tilde{\omega}\xi)[K_3(x_k,\overline{y}_l;\xi,\eta)+K_4(x_k,\overline{y}_l;\xi,\eta)]\,\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=\sum_{i=1}^{n}\sum_{j=-i}^{i-1}g_{ij}K_{ijkl}^{(3,4)} \tag{45}$$

DOI: 10.1002/fld

with

$$K^{(3,4)}_{ijkl} = \begin{cases} \dfrac{A_{lj}(\overline{y}_l - \overline{y}_j)^2}{\sqrt{1-\overline{y}_j^2}}[K_3(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j) + K_4(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j)]E_{ij}, & i \neq j \\ 0, & i = j \end{cases} \tag{46}$$

The kernels K_5, K_6, K_7 and K_8, have integrable singularities and we utilize the quadrature formulas

$$\iint_D^* \tilde{f}(\xi, \eta) \exp(\mathrm{i}\tilde{\omega}\xi) K_p(x_k, \overline{y}_l; \xi, \eta)\, \mathrm{d}\xi\, \mathrm{d}\eta = \sum_{i=1}^{n} \sum_{j=-i}^{i-1} g_{ij} K^{(p)}_{ijkl}, \quad p = 5, 6, 7, 8$$

where

$$K^{(5)}_{ijkl} = \begin{cases} \tilde{\omega}^2 B^{(k)}_{jl} E_{ij}, & i \leqslant k \\ 0, & i > k \end{cases} \tag{47}$$

$$B^{(k)}_{jl} = (y_{j+1} - \overline{y}_l) \ln|y_{j+1} - \overline{y}_l| - (y_j - \overline{y}_l) \ln|y_j - \overline{y}_l| \tag{48}$$

$$K^{(p)}_{ijkl} = E_{ij} K_p(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j)/n, \quad p = 6, 7, 8 \tag{49}$$

For calculating $K_7(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j)$, we use the series expansions of the Bessel and Strouve functions and we take into account that

$$K_7(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_l) = -\frac{\tilde{\omega}^2(\psi(1) + \psi(2))}{4} + \frac{\pi \mathrm{i}\tilde{\omega}}{4} \\ \psi(1) = -0.5772, \quad \psi(2) = 0.4228 \tag{50}$$

The kernel $K_8(x_k, \overline{y}_l; \overline{x}_{ij}, \overline{y}_j)$ is an integral which is evaluated numerically with the trapezoidal rule. Denoting

$$K_{ijkl} = K^{(1)}_{ijkl} + K^{(2)}_{ijkl} + K^{(3,4)}_{ijkl} + K^{(5)}_{ijkl} + K^{(6)}_{ijkl} + K^{(7)}_{ijkl} + K^{(8)}_{ijkl}$$

we obtain, discretizing the two-dimensional integral equation (2),

$$\frac{\varpi}{4\pi} \sum_{i=1}^{n} \sum_{j=-i}^{i-1} g_{ij} K_{ijkl} = -\left(\frac{\partial h(x_k, \overline{y}_l)}{\partial x} + \mathrm{i}\tilde{\omega} h(x_k, \overline{y}_l) \right) \exp(\mathrm{i}\tilde{\omega} x_k) \tag{51}$$

After solving this equation, we may obtain

$$\tilde{f}(x_k, \overline{y}_l) = \frac{g_{kl}}{\sqrt{x_k^2 - \overline{y}_l^2}}$$

4. THE AERODYNAMIC COEFFICIENTS

The pressure coefficient is

$$C_p(x^{(1)}, y^{(1)}, t) = Re[\tilde{f}(x, y)\ \exp(\mathrm{i}\omega t)] \tag{52}$$

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We are interested in the drag coefficient

$$C_{\mathrm{D}}(t) = -2\iint_D n_x C_p(ax, by, t)\,\mathrm{d}x\,\mathrm{d}y \tag{53}$$

We consider the oscillating delta wing whose equation is

$$0 = z^{(1)} - \alpha\exp(\mathrm{i}\omega_1 x^{(1)} + \mathrm{i}\omega t), \quad (x^{(1)}, y^{(1)}) \in D^{(1)} \tag{54}$$

where ω_1 is a constant complex number. Hence

$$h(x, y) = \alpha\exp(\mathrm{i}\widetilde{\omega}_1 x), \quad \widetilde{\omega}_1 = a\omega_1, \quad (x, y) \in D \tag{55}$$

For calculating the drag coefficient numerically, we employ the quadrature formulas

$$\begin{aligned} C_{\mathrm{D}}(t) = {}& \frac{2\alpha\,\mathrm{i}\tilde{\omega}_1\exp(\mathrm{i}\omega t)}{n^2}\cdot\sum_{k=1}^{n-1}\exp\left(\mathrm{i}\tilde{\omega}_1\frac{k}{n}\right)\\ &\times\left[\sum_{j=-k}^{k-1} C_p\left(\frac{ka}{n}, \frac{(2j+1)b}{2n}, t\right) + \frac{1}{2}C_p\left(\frac{ka}{n}, \frac{(-2k+1)b}{2n}, t\right)\right.\\ &\left. + \frac{1}{2}C_p\left(\frac{ka}{n}, \frac{(2k-1)b}{2n}, t\right)\right] \end{aligned} \tag{56}$$

5. THE LOW ASPECT RATIO OSCILLATORY DELTA WING. ANALYTICAL RESULTS

In this section, we consider that $\varpi \ll 1$. From (15)–(23) (we omit the details of calculus) it follows the asymptotic approximation

$$\int_{-\infty}^{x_0}\frac{\exp(\mathrm{i}\tilde{\omega}s)}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s = \frac{1}{\varpi^2 y_0^2}(1+\operatorname{sgn} x_0) + O(\ln(\varpi)) \tag{57}$$

whence we deduce that

$$\begin{aligned} &\iint_D^* \tilde{f}(\xi, \eta)\exp(-\mathrm{i}\tilde{\omega}x_0)\left(\int_{-\infty}^{x_0}\frac{\exp(\mathrm{i}\tilde{\omega}s)}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s\right)\mathrm{d}\xi\,\mathrm{d}\eta\\ &\quad = \iint_D^* \tilde{f}(\xi, \eta)\exp(-\mathrm{i}\tilde{\omega}x_0)\frac{1}{\varpi^2 y_0^2}(1+\operatorname{sgn} x_0)\,\mathrm{d}\xi\,\mathrm{d}\eta + O(\ln(\varpi)) \end{aligned} \tag{58}$$

We consider wings whose trailing edge is a straight line, perpendicular to Ox. In the Oxy-plane, we denote by

$$x = x_-(y) \tag{59}$$

or by

$$y = y_{\pm}(x) \tag{60}$$

the equations of the leading edge. We consider that the domain D is convex and the functions $y_+(x)$ and $y_-(x)$ are monotone. Introducing the function

$$P(x,\eta) = \int_{x_-(\eta)}^{x} \tilde{f}(\xi,\eta)\exp(-\mathrm{i}\tilde{\omega}x_0)\,\mathrm{d}\xi \tag{61}$$

from (57) and (58), we deduce

$$\iint_D^* \tilde{f}(\xi,\eta)\exp(-\mathrm{i}\tilde{\omega}x_0)\left(\int_{-\infty}^{x_0}\frac{\exp(\mathrm{i}\tilde{\omega}s)}{(s^2+\varpi^2 y_0^2)^{3/2}}\,\mathrm{d}s\right)\mathrm{d}\xi\,\mathrm{d}\eta$$

$$=2\mathrm{FP}\,{\int'}_{y_-(x)}^{y_+(x)} P(x,\eta)\frac{1}{\varpi^2 y_0^2}\,\mathrm{d}\eta + O(\ln\varpi) \tag{62}$$

where

$$\mathrm{FP}\int_{y_-(x)}^{y_+(x)} P(x,\eta)\frac{1}{y_0^2}\,\mathrm{d}\eta = \lim_{\varepsilon\to 0}\left[\left(\int_{y_-(x)}^{y-\varepsilon}+\int_{y+\varepsilon}^{y_+(x)}\right)P(x,\eta)\frac{1}{y_0^2}\,\mathrm{d}\eta - 2\frac{P(x,y)}{\varepsilon}\right] \tag{63}$$

is the finite part of the integral taken into consideration in Ch. Fox's sense [12]. From (2) and (62), we deduce that the two-dimensional integral equation is reduced to

$$\frac{1}{2\pi}\mathrm{FP}\int_{y_-(x)}^{y_+(x)}\frac{P(x,\eta)}{y_0^2}\,\mathrm{d}\eta = -\varpi\left(\frac{\partial h}{\partial x}(x,y)+\mathrm{i}\tilde{\omega}h(x,y)\right) \tag{64}$$

if we neglect the terms of order $O(\varpi^2\ln\varpi)$. After solving (64), we find $\tilde{f}(x,y)$ using the relation

$$\tilde{f}(x,y) = \frac{\partial P(x,y)}{\partial x} + \mathrm{i}\tilde{\omega}P(x,y) \tag{65}$$

Since

$$\frac{1}{y_0^2} = -\frac{\mathrm{d}}{\mathrm{d}y}\frac{1}{y_0} = \frac{\mathrm{d}}{\mathrm{d}\eta}\frac{1}{y_0}$$

and

$$P(x,y_-(x)) = P(x,y_+(x)) = 0 \tag{66}$$

after an integration by parts, (64) becomes

$$\frac{1}{2\pi}{\int'}_{y_-(x)}^{y_+(x)}\frac{\partial P(x,\eta)}{\partial\eta}\frac{\mathrm{d}\eta}{y_0} = \varpi\left(\frac{\partial h}{\partial x}(x,y)+\mathrm{i}\tilde{\omega}h(x,y)\right) \tag{67}$$

where the prime '$'$' is for Cauchy's principal value. From (66), we get

$$\int_{y_-(x)}^{y_+(x)}\frac{\partial P(x,\eta)}{\partial\eta}\,\mathrm{d}\eta = 0 \tag{68}$$

The solution of the integral singular equation (67) with the condition (68) is [10]:

$$\frac{\partial P(x,\eta)}{\partial \eta} = \frac{2}{\pi}\frac{\varpi}{\sqrt{(\eta-y_-)(y_+-\eta)}}\int_{y_-(x)}^{\prime y_+(x)} \frac{\sqrt{(t-y_-)(y_+-t)}}{t-\eta}\left(\frac{\partial h}{\partial x}(x,t)+\mathrm{i}\tilde{\omega}h(x,t)\right)\mathrm{d}t$$

whence

$$P(x\cdot y) = \frac{2\varpi}{\pi}\int_{y_-}^{y}\left(\int_{y_-}^{\prime y_+}\frac{\sqrt{(t-y_-)(y_+-t)}}{t-\eta}\left(\frac{\partial h(x,t)}{\partial x}+\mathrm{i}\tilde{\omega}h(x,t)\right)\mathrm{d}t\right)\frac{\mathrm{d}\eta}{\sqrt{(\eta-y_-)(y_+-\eta)}} \quad (69)$$

In the particular case when h does not depend on the second variable, on the basis of relation

$$\int_{y_-}^{\prime y_+}\frac{\sqrt{(t-y_-)(y_+-t)}}{t-\eta}\,\mathrm{d}t = \pi\left(\frac{y_++y_-}{2}-\eta\right) \quad (70)$$

we deduce

$$\int_{y_-}^{y}\left(\int_{y_-}^{\prime y_+}\frac{\sqrt{(t-y_-)(y_+-t)}}{t-\eta}\mathrm{d}t\right)\frac{\mathrm{d}\eta}{\sqrt{(\eta-y_-)(y_+-\eta)}} = \pi\sqrt{(y-y_-)(y_+-y)} \quad (71)$$

whence it follows

$$P(x,y) = 2\varpi\sqrt{(y-y_-)(y_+-y)}\cdot\left(\frac{\mathrm{d}h}{\mathrm{d}x}+\mathrm{i}\tilde{\omega}h(x)\right) \quad (72)$$

Taking into account (65) we deduce

$$\tilde{f}(x,y) = 2\varpi\frac{\partial}{\partial x}\left[\left(\frac{\mathrm{d}h}{\mathrm{d}x}+\mathrm{i}\tilde{\omega}h(x)\right)\sqrt{(y-y_-)(y_+-y)}\right] + 2\mathrm{i}\tilde{\omega}\varpi\left(\frac{\mathrm{d}h}{\mathrm{d}x}+\mathrm{i}\tilde{\omega}h(x)\right)\sqrt{(y-y_-)(y_+-y)} \quad (73)$$

For the delta plate, with

$$h(x,y) = \alpha\exp(\mathrm{i}\tilde{\omega}_1 x), \quad y_+(x) = x, \quad y_-(x) = -x, \quad x\in[0,1]$$

from (73), we obtain

$$\tilde{f}(x,y) = 2\alpha\varpi\exp(\mathrm{i}\tilde{\omega}_1 x)\left[\mathrm{i}(\tilde{\omega}_1+\tilde{\omega})\frac{x}{\sqrt{x^2-y^2}} - (\tilde{\omega}_1+\tilde{\omega})^2\sqrt{x^2-y^2}\right] \quad (74)$$

From the previous formula, we deduce that

$$\tilde{f}(x,y) = \frac{g(x,y)}{\sqrt{x^2-y^2}} \quad (75)$$

where $g(x,y)$ is a finite function.

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6. ANALYTICAL AND NUMERICAL RESULTS. THE PROPULSIVE FORCE

In order to have an intuitive view of the temporal variation of the pressure coefficient distribution over the wing, in Figures 1 and 2, we present the shape of the flexible wing and the pressure coefficient field C_P/α for the values of the non-dimensional time $2\omega t/\pi \in \{0, 1, 2, 3\}$ and for the non-dimensional frequencies $\tilde{\omega} = \pi/4$, $\widetilde{\omega}_1 = 2\pi/3$.

We considered $n = 15$ (i.e. 225 nodes) and $\varpi = 1/10$. In Figure 1, the pressure coefficient field was obtained with the numerical method and in Figure 2 it was obtained by means of analytical formula (74). We notice that the corresponding coefficient pressure fields from the two figures are very close. Another visual comparison between the numerical and analytical results is performed in Figure 3. In the right-hand column, we present the numerical results. For every value of the non-dimensional time $2\omega t/\pi \in \{0, 3/2, 3\}$, one draws a wireframe mesh and a contour plot beneath the mesh for the pressure coefficient field C_P/α *versus* the nodes of the grid. Similar analytical results are presented in the left-hand column.

We notice that the numerical and analytical results are close but not identical. The main reason for the existence of some differences between the two types of results is that the analytical solution is the solution of an integral equation which approximates the original integral equation. Since the

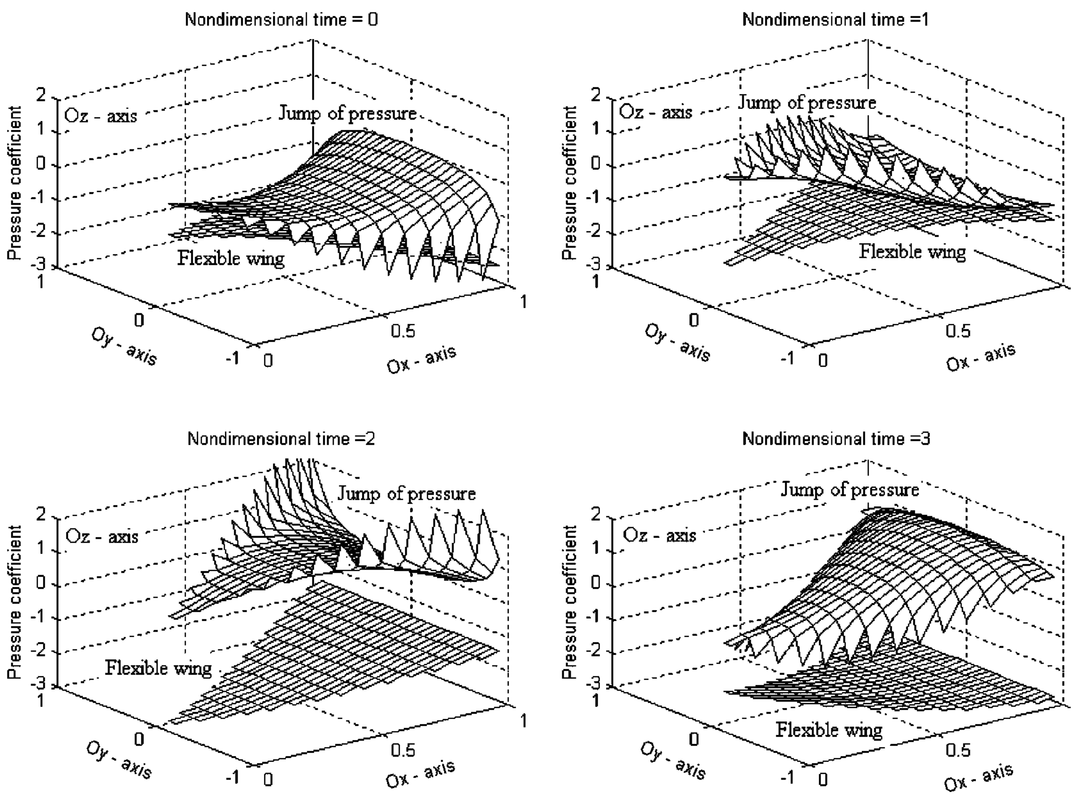

Figure 1. Pressure coefficient field calculated anatically.

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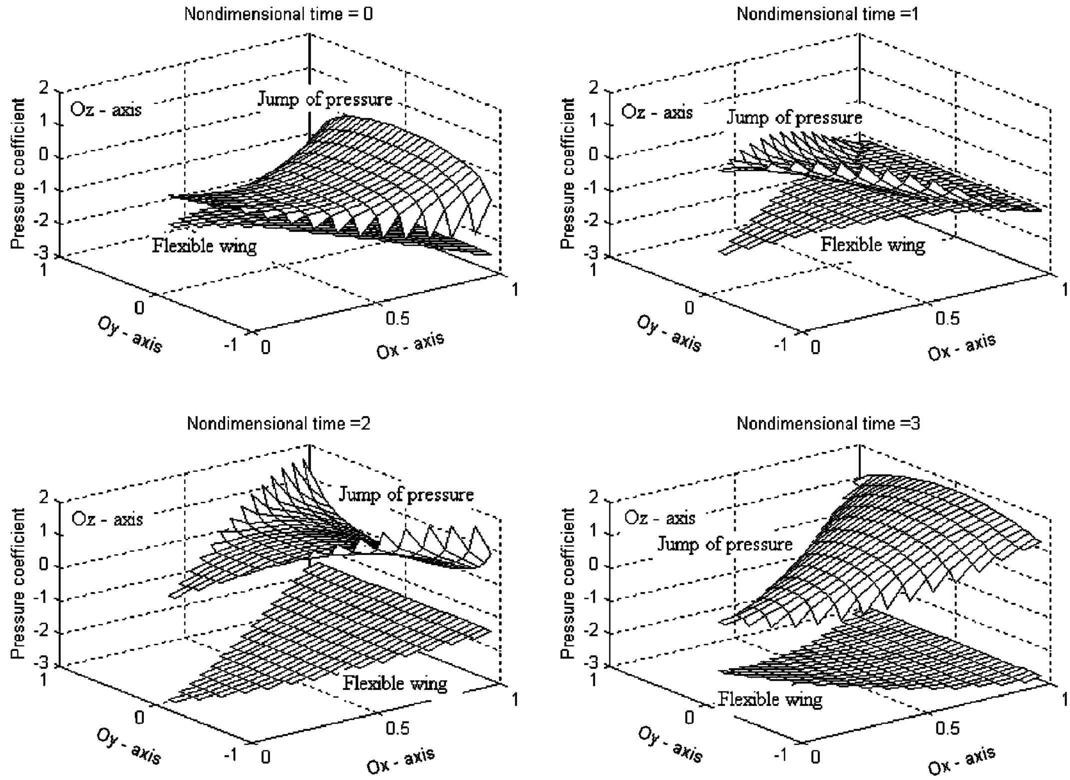

Figure 2. Pressure coefficient field calculated numerically.

order of approximation is $O(\varpi \ln \varpi)$ we expect to have a better agreement between the analytical and numerical results when ϖ decreases. In order to verify that the numerical solutions of the integral equations converge to the exact solution we may check up that the difference between two solutions corresponding to two different numbers of nodes (control points) decreases when the two different numbers of nodes increase.

In Figure 4, we present the pressure coefficient distribution C_P/α along the trailing edge of the wing for the aspect ratio $\varpi = 1/2$, $\varpi = 1/10$ and $\varpi = 1/100$. We notice that for small values of ϖ, the numerical results come nearer to the analytical ones. The numerical results are calculated for 25, 100, respectively, 225 nodes. We find that in the points from the trailing edge the difference between the pressure coefficients calculated for 225 and 100 grid nodes is smaller than the difference between the pressure coefficients calculated for 25 and 100 grid nodes.

Since the drag coefficient varies periodically, we may calculate the *average drag coefficient*

$$\widetilde{C}_\mathrm{D} = \frac{1}{T}\int_0^T C_\mathrm{D}(t)\,\mathrm{d}t \simeq \frac{1}{k}\sum_{l=1}^{k} C_\mathrm{D}\left(\frac{lT}{k}\right)$$

where T is the period of the oscillation.

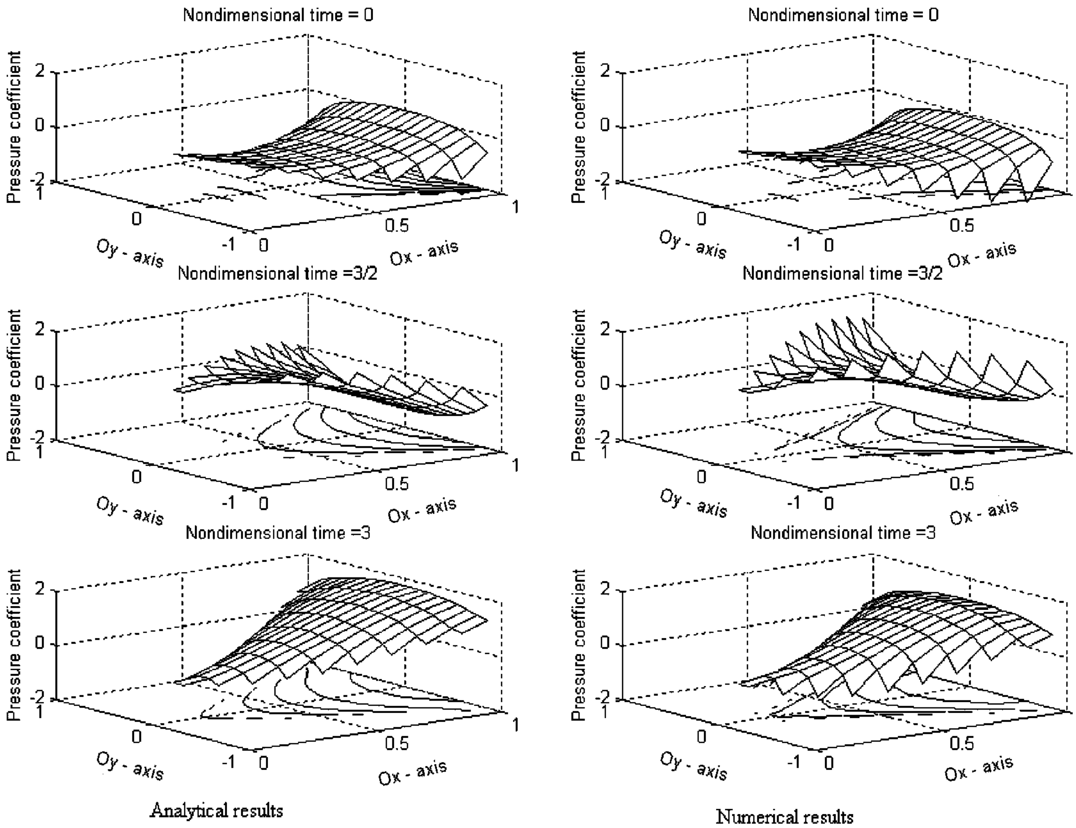

Figure 3. Comparison between pressure coefficient fields calculated numerically and analytically.

In Figure 5 we present, for $\widetilde{\omega}_1 = \pi/4$, the average drag coefficient (divided by α^2) *versus* the reduced frequency $\widetilde{\omega}$. We notice that the average drag coefficient is negative, i.e. it appears a propulsive force (thrust) and the absolute value of this propulsive force increases when $\widetilde{\omega}$ increases. In fact there exists a drag, $F>0$ determined for example by the viscosity of the fluid (that we have not taken into consideration). Assume that for a certain critical value $\widetilde{\omega}_{cr}$ of the reduced frequency we have

$$F + F_D = 0 \tag{76}$$

where $F_D = \frac{1}{2}\rho_0 V_0^2 ab\widetilde{C}_D$ is the propulsive force. Condition (76) is necessary for having an average constant translation velocity V_0 of the wing. When $\widetilde{\omega} > \widetilde{\omega}_{cr}$, we have $F + F_D < 0$ and it appears an acceleration having the sense and direction of the uniform translation velocity, which increases. In the same time $\widetilde{\omega} = \omega a/V_0$ diminishes until it attains the value $\widetilde{\omega}_{cr}$. Conversely, when $\widetilde{\omega} < \widetilde{\omega}_{cr}$, we have $F + F_D < 0$ and it appears an acceleration having the sense opposite to the uniform translation velocity, which decreases. In the same time $\widetilde{\omega} = \omega a/V_0$ increases until it attains the value $\widetilde{\omega}_{cr}$. Hence, the average velocity of the wing tends to become constant. We conclude that for obtaining the null drag, to every value ω of the frequency, it corresponds a value V_0 of the uniform translation velocity and V_0 is an increasing function of ω.

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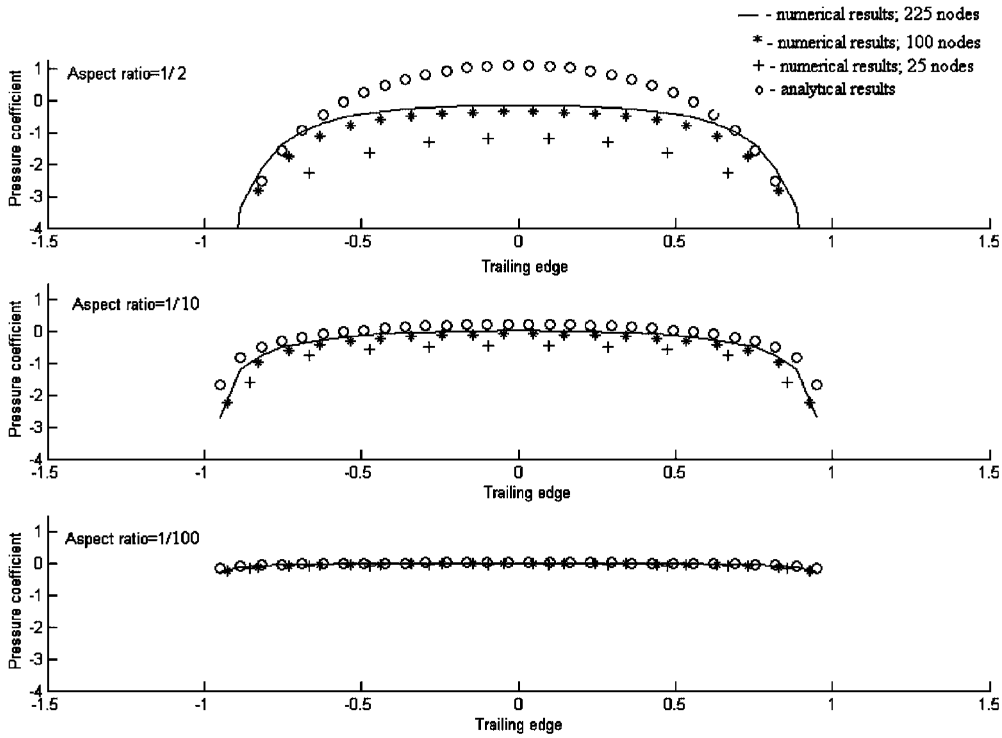

Figure 4. Pressure coefficient field on the trailing edge.

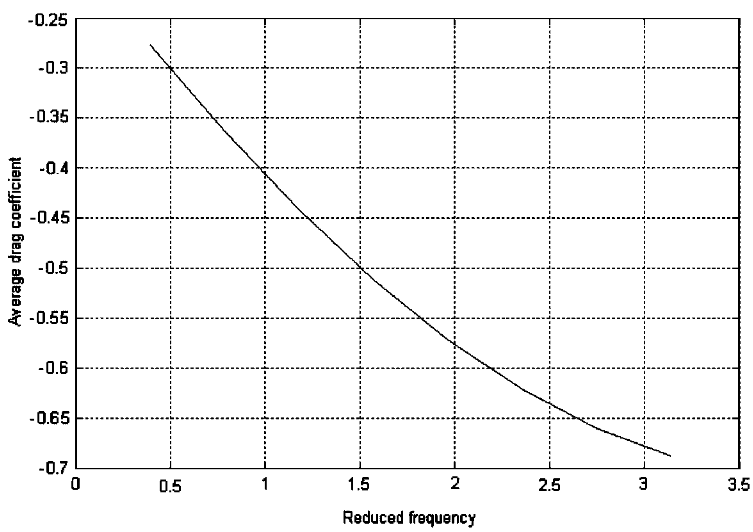

Figure 5. Average drag coefficient *versus* reduced frequency.

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NOMENCLATURE

$Ox^{(1)}y^{(1)}z^{(1)}$	system of Cartesian coordinates moving with the wing
$Oxyz$	fixed system of Cartesian coordinates
t	time
a	wing chord
b	half-span
$(x,y,z)=(x^{(1)}/a, y^{(1)}/b, z^{(1)}/a)$	dimensionless Cartesian coordinates
$D^{(1)}$	wing projection on the $Ox^{(1)}y^{(1)}$-plane
$D=\{(x,y);(ax,by)\in D^{(1)}\}$;	
V_0	uniform translation velocity of the wing
$\varpi=b/a$	aspect ratio
ω	frequency of the oscillation
$\widetilde{\omega}=\omega a/V_0$	reduced frequency
$z^{(1)}-h^{(1)}(x^{(1)},y^{(1)})\exp(\mathrm{i}\omega t)$	equation of the oscillating wing
ρ_0	density of the fluid at rest
$\mathbf{v}=(u,v,w)$	velocity
p	pressure
C_p	pressure coefficient
C_{D}	drag coefficient
$\widetilde{C}_{\mathrm{D}}$	average drag coefficient
$\rho_0 Re(f\exp(\mathrm{i}\omega t))$	jump of the pressure
$h(x,y)=\frac{h^{(1)}(x^{(1)},y^{(1)})}{a},\ \widetilde{f}(x,y)=\frac{f(x^{(1)},y^{(1)})}{V_0^2}$	dimensionless functions
ξ,η	dimensionless Cartesian variables
$x_0=x-\xi,\ y_0=y-\eta$;	
I_1, K_1	Bessel functions
L_{-1}	Strouve function
α,ω_1	complex constants
$z^{(1)}=\alpha\exp(\mathrm{i}\omega_1 x^{(1)}+\mathrm{i}\omega t)$	equation of the oscillating delta wing
$K(x,y;\xi,\eta),\ldots,K_8(x,y;\xi,\eta)$	kernels of the integral equation.

APPENDIX A

We consider the continuity and Euler equations for incompressible flow in a fixed Cartesian frame of reference $Oxyz$,

$$\begin{gathered}\operatorname{div}\mathbf{v}=0,\quad \mathbf{v}=(u,v,w)\\ \frac{\partial\mathbf{v}}{\partial t}+(\mathbf{v}\cdot\operatorname{grad})\mathbf{v}+\frac{1}{\rho_0}\operatorname{grad}p=0\end{gathered}\tag{A1}$$

Writing the equation of the wing

$$S: F(x,y,z,t)=z-h(x,y,t),\quad (x,y)\in D$$

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we have the coordinates of the normal at the wing surface S:

$$\mathbf{n}=(n_x,n_y,n_z,n_t)=\left(-\frac{\partial h}{\partial x},-\frac{\partial h}{\partial y},1,-\frac{\partial h}{\partial t}\right) \tag{A2}$$

We linearize the equations around the rest state (neglecting the products of the perturbation quantities) and we write them into distributions:

$$\frac{\partial u}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial x}=\left([u]_S n_t+\frac{1}{\rho_0}[p]_S n_x\right)\delta_S=0 \tag{A3}$$

$$\frac{\partial v}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial y}=\left([v]_S n_t+\frac{1}{\rho_0}[p]_S n_y\right)\delta_S=0 \tag{A4}$$

$$\frac{\partial w}{\partial t}+\frac{1}{\rho_0}\frac{\partial p}{\partial z}=\left([w]_S n_t+\frac{1}{\rho_0}[p]_S n_z\right)\delta_S=f\delta_{\mathrm{D}},\quad f=\frac{[p]_S}{\rho_0} \tag{A5}$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\left([u]_S n_x+[v]_S n_y+[w]_S n_z\right)\delta_S=0 \tag{A6}$$

where $\mu\delta_S$ represents the simple layer distribution with density μ and $[\cdot]_S$ is the jump over the surface S. From (A3)–(A6) we get

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)p=\frac{\partial}{\partial z}([p]_S\delta_{\mathrm{D}})=\frac{\partial}{\partial z}(\rho_0 f\delta_{\mathrm{D}}) \tag{A7}$$

Since

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(-\frac{\delta(t)}{4\pi|\mathbf{x}|}\right)=\delta(x,y,z)\delta(t),\quad |\mathbf{x}|=\sqrt{x^2+y^2+z^2} \tag{A8}$$

we get

$$p=-\rho_0\frac{\delta(t)}{4\pi}\frac{\partial}{\partial z}\frac{1}{|\mathbf{x}|}*f\delta_{\mathrm{D}} \tag{A9}$$

whence, taking into account (A3), we deduce

$$\frac{\partial w}{\partial t}=\frac{\delta(t)}{4\pi}\frac{\partial}{\partial z^2}\frac{1}{|\mathbf{x}|}*f\delta_{\mathrm{D}}+f\delta_{\mathrm{D}} \tag{A10}$$

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From (A9) and (A10) we have, denoting by $H(t)$ Heaviside's function,

$$\begin{aligned} w(x,y,z,t) &= -\frac{H(t)}{4\pi}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{1}{|\mathbf{x}|} * f\delta_{\mathrm{D}} \\ &= -\frac{1}{4\pi}\int_{-\infty}^{\infty} H(t-t')\,\mathrm{d}t' \iint_{D(t')}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\frac{f(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}x'\,\mathrm{d}y' \\ &\overset{z\neq 0}{=} \frac{1}{4\pi}\int_{-\infty}^{t}\iint_{D(t')}\frac{\partial^2}{\partial z^2}\frac{f(\mathbf{x}',t)}{|\mathbf{x}-\mathbf{x}'|}\,\mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}t' \end{aligned} \tag{A11}$$

We introduce a new system of coordinates $O^{(1)}x^{(1)}y^{(1)}z^{(1)}$ which is related to the lifting surface and has a uniform translation motion with the velocity $-V_0\mathbf{i}$ with respect to the $Oxyz$ system of coordinates. We have the relations

$$x^{(1)} = x + V_0 t, \quad y^{(1)} = 1, \quad z^{(1)} = z \tag{A12}$$

Denoting $x'^{(1)} = x' + V_0 t'$, $s^{(1)} = x^{(1)} - x'^{(1)} - V_0(t-t')$, integral representation (A11) becomes, for $z^{(1)} \neq 0$,

$$w(\mathbf{x}^{(1)},t) = \frac{1}{4\pi V_0}\int_{-\infty}^{x^{(1)}-x'^{(1)}} \mathrm{d}s^{(1)} \int_{D^{(1)}} \frac{\partial^2}{\partial z^{(1)2}} \frac{f(\mathbf{x}'^{(1)},t)}{\sqrt{s^{(1)2}+(y^{(1)}-y'^{(1)})^2+z^{(1)2}}}\,\mathrm{d}x'^{(1)}\,\mathrm{d}y'^{(1)} \tag{A13}$$

where $D^{(1)}$, the projection of the lifting surface onto the $O^{(1)}x^{(1)}y^{(1)}$-plane is a fixed surface.

Considering the lifting surface subjected to harmonic oscillations, we set

$$w(\mathbf{x}^{(1)},t) = d^{(1)}(\mathbf{x}^{(1)})\exp(\mathrm{i}\omega t), \quad f(x^{(1)},y^{(1)},t) = f(x^{(1)},y^{(1)})\exp(\mathrm{i}\omega t) \tag{A14}$$

whence it follows

$$\begin{aligned} &\iint_{D^{(1)}} \mathrm{d}x'^{(1)}\,\mathrm{d}y'^{(1)} \int_{-\infty}^{x^{(1)}-x'^{(1)}} f(x'^{(1)},y'^{(1)})\exp\left(-\mathrm{i}\frac{\omega}{V_0}\left(x^{(1)}-x'^{(1)}-s^{(1)}\right)\right) \\ &\quad\times \frac{\partial^2}{\partial z^{(1)2}} \frac{1}{\sqrt{s^{(1)2}+(y^{(1)}-y'^{(1)})^2+z^{(1)2}}}\,\mathrm{d}s^{(1)} = 4\pi V_0 d^{(1)}(\mathbf{x}^{(1)}) \end{aligned} \tag{A15}$$

We introduce the dimensionless coordinates

$$(x,y,z,s,\xi\eta) = \left(\frac{x^{(1)}}{a},\frac{y^{(1)}}{b},\frac{z^{(1)}}{a},\frac{s^{(1)}}{a},\frac{x'^{(1)}}{a},\frac{y^{(1)}}{b}\right) \tag{A16}$$

For the sake of simplicity, we use again the notation (x, y, z) which must not be confounded with the notations for the variables of the fixed system $Oxyz$. Denoting

$$D = \{(x,y);\,(ax,by)\in D^{(1)}\}$$

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and passing to limit for $z \to 0$, we get from (A15)

$$\frac{ab}{4\pi V_0} \iint_D^* f(a\xi, b\eta) \exp\left(-\mathrm{i}\frac{\omega}{V_0} a(x-\xi)\right) \left(\int_{-\infty}^{a(x-\xi)} \frac{\exp\left(\mathrm{i}\dfrac{\omega}{V_0} s\right) \mathrm{d}s}{(s^2 + b^2(y-\eta)^2)^{3/2}} \right) \mathrm{d}\xi \, \mathrm{d}\eta$$

$$= -d(x, y) \tag{A17}$$

In the framework of the linearized theory,

$$d(x, y) \exp(\mathrm{i}\omega t) = w(x^{(1)}, y^{(1)}, t) \tag{A18}$$

where w is the projection of the velocity on the $Oz^{(1)}$-axis.

The velocity field with respect to the $Ox^{(1)}y^{(1)}z^{(1)}$-system of coordinates is

$$\mathbf{V} = V_0 \mathbf{i} + \mathbf{v}, \quad \mathbf{v} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$

where $\mathbf{v}$ is the perturbation velocity of the fluid.

For calculating the downwash distribution, we employ the slipping condition

$$\mathbf{V} \cdot \mathbf{n}|_{D^{(1)}} = -\frac{\partial F / \partial t}{|\mathrm{grad}\, F|} \tag{A19}$$

with

$$\mathbf{n} = \frac{\mathrm{grad}\, F}{|\mathrm{grad}\, F|} = -\frac{\partial h^{(1)}}{\partial x^{(1)}} \exp(\mathrm{i}\omega t)\mathbf{i} - \frac{\partial h^{(1)}}{\partial y^{(1)}} \exp(\mathrm{i}\omega t)\mathbf{j} + \mathbf{k} \tag{A20}$$

Since

$$\frac{\partial F}{\partial t} = -\mathrm{i}\omega h^{(1)}(x^{(1)}, y^{(1)}) \exp(\mathrm{i}\omega t) \tag{A21}$$

from (A19) and (A20), we obtain the linearized condition

$$w = \left(V_0 \frac{\partial h^{(1)}}{\partial x^{(1)}} + \mathrm{i}\omega h^{(1)} \right) \exp(\mathrm{i}\omega t) \tag{A22}$$

Denoting

$$h(x, y) = \frac{h^{(1)}(x^{(1)}, y^{(1)})}{a}, \quad \tilde{\omega} = \frac{\omega a}{V_0}$$

from (A18) and (A22) it follows

$$d(x, y) = V_0 \left(\frac{\partial h(x, y)}{\partial x} + \mathrm{i}\tilde{\omega} h(x, y) \right) \tag{A23}$$

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Introducing the dimensionless functions and variables

$$\tilde{d} = \frac{d}{V_0}, \quad \tilde{f}(x, y) = \frac{f(ax, by)}{V_0^2}, \quad x_0 = x - \xi, \quad y_0 = y - \eta$$

equation (A17) becomes

$$\frac{\varpi}{4\pi} \iint_D^* \tilde{f}(\xi, \eta) \exp(-\mathrm{i}\tilde{\omega}x_0) \left(\int_{-\infty}^{x_0} \frac{\exp(\mathrm{i}\tilde{\omega}s)}{(s^2 + \varpi^2 y_0^2)^{3/2}} \,\mathrm{d}s \right) \mathrm{d}\xi \,\mathrm{d}\eta$$

$$= -\left(\frac{\partial h(x, y)}{\partial x} + \mathrm{i}\tilde{\omega} h(x, y) \right) \tag{A24}$$

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